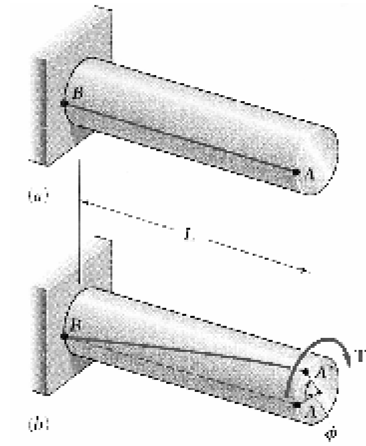
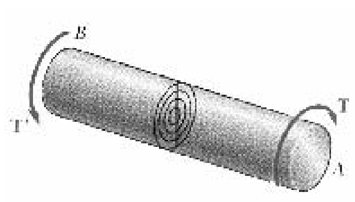
Torsion, Thermal Effects and Indeterminacy

Deformation in Torsionally Loaded Members



Axi-symmetric cross sections subjected to axial moment or **torque** will remain plane and undistorted.

At a section, internal torque (resisiting applied torque) is made up of shear forces parallel to the area and in the direction of the torque. The distribution of the shearing stresses depends on the angle of twist, ϕ. The cross section remains plane and undistored.



Shearing Strain

Shearing strain is the angle change of a straight line segment along the axis.

$$\gamma = \frac{\rho\phi}{L}$$

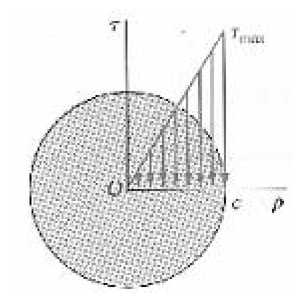
where

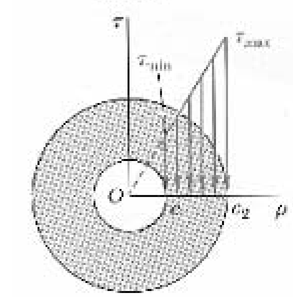
ρ is the radial distance from the centroid to the point under strain.

The maximum strain is at the surface, a distance c from the centroid: $\gamma_{max} = \dfrac{c\phi}{L}$

G is the <u>Shear Modulus</u> or <u>Modulus of Rigidity:</u> $\tau = G \cdot \gamma$

Shearing Strain and Stress





In the linear elastic range: the torque is the summation of torsion stresses over the area:

$$T = \frac{\tau J}{\rho} \qquad \text{gives:} \qquad \tau = \frac{T\rho}{J}$$

Maximum torsional stress, τ_{max}, occurs at the **outer diameter** (or **perimeter**).

Polar Moment of Inertia

For axi-symmetric shapes, there is only one value for polar moment of inertia, J, determined by the radius, c:

solid section: $J = \dfrac{\pi c^4}{2}$ hollow section: $J = \dfrac{\pi\left(c_o^{\,4} - c_i^{\,4}\right)}{2}$

Combined Torsion and Axial Loading

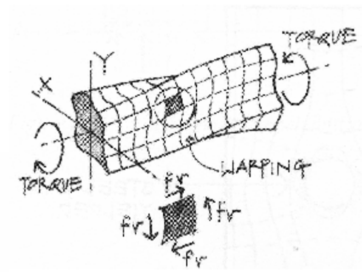
Just as with combined axial load and shear, combined torsion and axial loading result in maximum shear stress at a 45° oblique "plane" of twist.

Shearing Strain

In the linear elastic range: $\phi = \dfrac{TL}{JG}$ and for composite shafts: $\phi = \sum_i \dfrac{T_i L_i}{J_i G_i}$

Torsion in Noncircular Shapes

J is no longer the same along the lateral axes. Plane sections do not remain plane, but distort. τ_{max} is still at the furthest distance away from the centroid. For rectangular shapes:



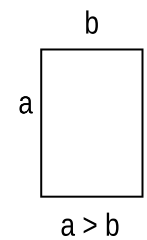
$$\tau_{max} = \frac{T}{c_1 ab^2} \qquad \phi = \frac{TL}{c_2 ab^3 G}$$

For a/b > 5:

$$c_1 = c_2 = \frac{1}{3}\left(1 - 0.630\, b/a\right)$$

TABLE 3.1. Coefficients for Rectangular Bars in Torsion

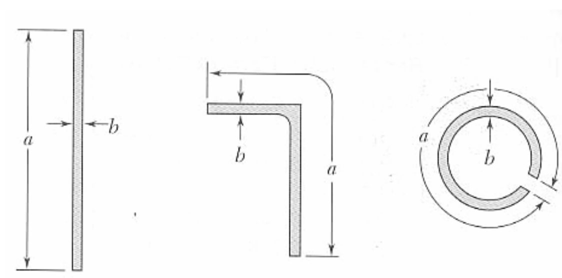
a/b	c_1	c_2
1.0	0.208	0.1406
1.2	0.219	0.1661
1.5	0.231	0.1958
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333



Open Sections

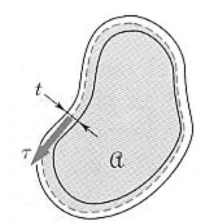
For long narrow shapes where a/b is very large ($a/b \rightarrow \infty$) $c_1 = c_2 = 1/3$ and:

$$\tau_{max} = \frac{T}{\frac{1}{3} ab^2} \qquad \phi = \frac{TL}{\frac{1}{3} ab^3 G}$$



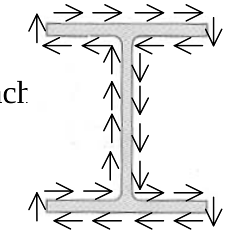
Shear Flow of Closed Thin Walled Sections

q is the internal shearing force per unit length, and is constant on a cross section even though the thickness of the wall may very. $\mathcal{a}$ is the area bounded by the centerline of the wall section; s_i, is a length segment of the wall and t_i is the corresponding thickness of the length segment.



$$\tau = \frac{T}{2t\mathcal{a}} \qquad \phi = \frac{TL}{4t\mathcal{a}^2}\sum_i \frac{s_i}{t_i}$$

Shear Flow in Open Sections



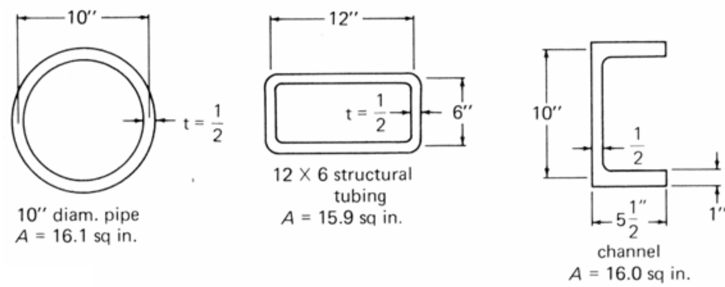
The shear flow must wrap around at all edges, and the total torque is distributed among the areas making up the cross section in proportion to the torsional rigidity of each rectangle ($ab^2/3$). The total angle of twist is the sum of the ϕ values from each rectangle. t_i is the thickness of each rectangle and b_i is the length of each rectangle.

$$\tau_{max} = \frac{Tt_{max}}{\frac{1}{3}\Sigma b_i t_i^3} \qquad \phi = \frac{TL}{\frac{1}{3}G\Sigma b_i t_i^3}$$

<u>Example 1</u>

Example 8.9.1

Compare the torsional resisting moment T and the torsional constant J for the sections of Fig. 8.9.4 all having about the same cross-sectional area. The maximum shear stress τ is 14 ksi.



SOLUTION

(a) Circular thin-wall section.

$$T = \frac{\tau J}{\rho} = \frac{(14ksi)(393.7in^4)}{5.25in}\cdot\frac{1ft}{12in} = 87.5k-ft$$

$$J = \frac{\pi(c_o^4 - c_i^4)}{2} = \frac{\pi((5.25in)^4 - (4.75in)^4)}{2} = 393.7in^4$$

(b) Rectangular box section. $\tau = \dfrac{T}{2t\mathcal{a}}$

$$T = \tau 2t\mathcal{a} = (14ksi)2(0.5in)(72in^2)\cdot\frac{1ft}{12in} = 84k-ft$$

$$\mathcal{a} \approx (12in)(6in) = 72in^2$$

(c) Channel section. Since for this open section,

$$\tau_{max} = \frac{Tt_{max}}{\frac{1}{3}\sum b_i t_i^3} = \frac{Tt}{J} \qquad T = \frac{\tau J}{t_{max}}\frac{(14ksi)(4.08in^4)}{1in}\cdot\frac{1ft}{12in} = 4.8k-ft$$

the maximum shear stress will be in the flange. Also,

$$J = \sum\frac{bt^3}{3} \qquad J = \tfrac{1}{3}\left[10in(0.5in)^3 + (5.5in)(1in)^3 + (5.5in)(1in)^3\right] = 4.08in^4$$

Thermal Strains

Physical restraints limit deformations to be the same, or sum to **zero**, or be proportional with respect to the rotation of a rigid body.

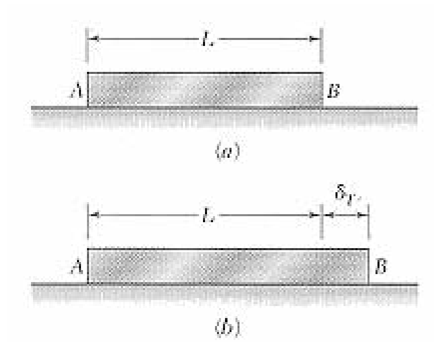
We know axial stress relates to axial strain: $\delta = \dfrac{PL}{AE}$ which relates δ to P

Deformations can be caused by the *material* reacting to a change in energy with temperature. In general (there are some exceptions):

- Solid materials can **contract** with a decrease in temperature.
- Solid materials can **expand** with an increase in temperature.

The change in length per unit temperature change is the *coefficient of thermal expansion*, α. It has units of $/^{\circ}F$ or $/^{\circ}C$ and the deformation is related by: $\delta_T = \alpha(\Delta T)L$

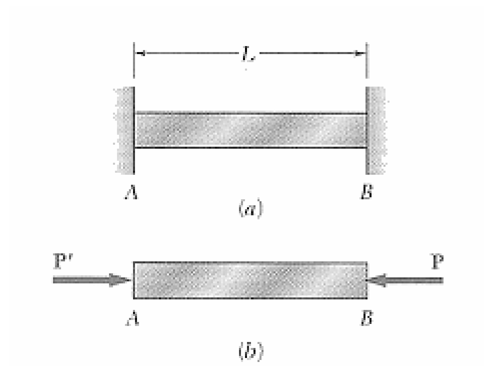
Thermal Strain: $\varepsilon_T = \alpha\Delta T$



There is **no stress** associated with the length change with free movement, BUT if there are restraints, thermal deformations or strains *can cause internal forces and stresses.*

How A Restrained Bar Feels with Thermal Strain

1. Bar pushes on supports because the material needs to expand with an increase in temperature.
2. Supports push *back*.
3. Bar is restrained, can't move and the reaction causes internal *stress*.



Superposition Method

If we want to solve a statically indeterminate problem that has extra support forces:

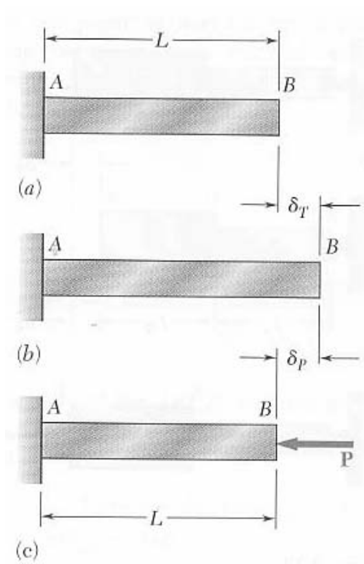
- We can remove a support or supports that *makes the problem look statically determinate*
- Replace it with a reaction and treat it like it is an applied force
- Impose geometry restrictions that the support imposes

For Example:

$\delta_T = \alpha(\Delta T)L$ $\qquad$ $\delta_p = -\dfrac{PL}{AE}$

$\delta_P + \delta_T = 0$ $\qquad$ $-\dfrac{PL}{AE} + \alpha(\Delta T)L = 0$

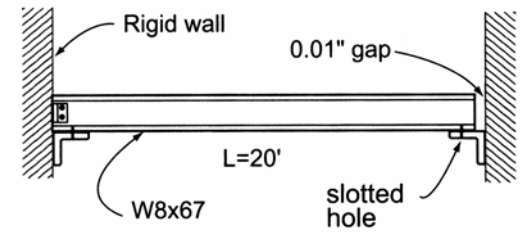
$P = \alpha(\Delta T)L\dfrac{AE}{L} = \alpha(\Delta T)AE$ $\qquad$ $f = -\dfrac{P}{A} = -\alpha(\Delta T)E$



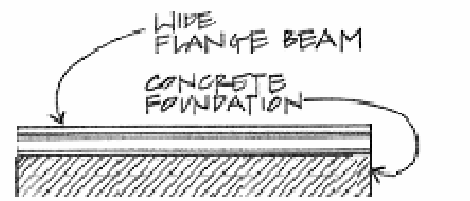
Example 2 (pg 228)

Example Problem 6.24 (Figures 6.58 and 6.59)

A W8×67 steel beam, 20 ft. in length, is rigidly attached at one end of a concrete wall. If a gap of 0.010 in. exists at the opposite end when the temperature is 45°F, what results when the temperature rises to 95°F?

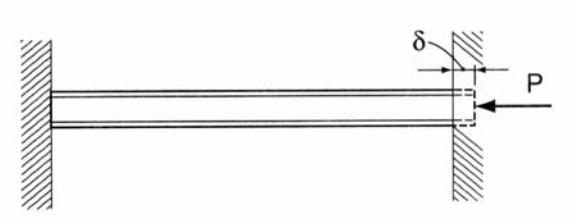


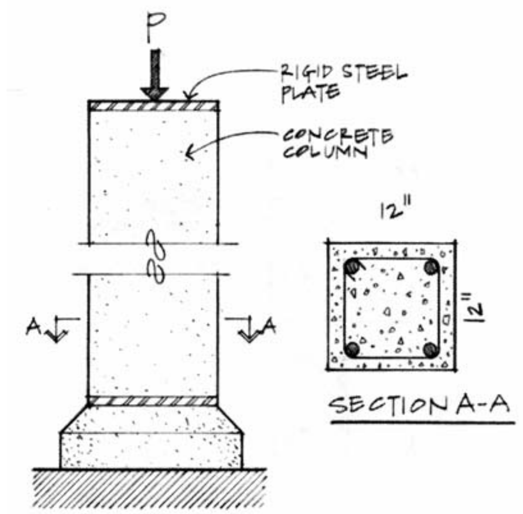
ALSO: If the beam is anchored to a concrete slab, and the steel sees a temperature change of 50° F while the concrete only sees a change of 30° F, determine the compressive stress in the beam.

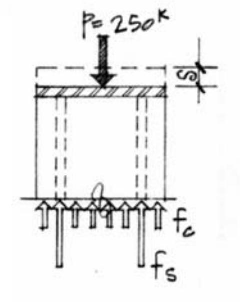


$\alpha_c = 5.5 \times 10^{-6}$ /° F $\qquad E_c = 3 \times 10^6$ psi

$\alpha_s = 6.5 \times 10^{-6}$ /° F $\qquad E_s = 29 \times 10^6$ psi







Example 3

5.21 A short concrete column measuring 12 in. square is reinforced with four #8 bars ($A_s = 4 \times 0.79$ in.^{2} = 3.14 in.^{2}) and supports an axial load of 250k. Steel bearing plates are used top and bottom to ensure equal deformations of steel and concrete. Calculate the stress developed in each material if:

$E_c = 3 \times 10^6$ psi and

$E_s = 29 \times 10^6$ psi

Solution:

From equilibrium:

$$[\Sigma F_y = 0] - 250\text{ k} + f_s A_s + f_c A_c = 0$$

$$A_s = 3.14\text{ in.}^2$$

$$A_c = (12'' \times 12'') - 3.14\text{ in.}^2 \cong 141\text{ in.}^2$$

$$3.14 f_s + 141 f_c = 250\text{ k}$$

From the deformation relationship:

$$\delta_s = \delta_c;\ L_s = L_c$$

$$\therefore \frac{\delta_s}{L} = \frac{\delta_c}{L}$$

and

$$\varepsilon_s = \varepsilon_c$$

Since

$$E = \frac{f}{\varepsilon}$$

and

$$\frac{f_s}{E_s} = \frac{f_c}{E_c}$$

$$f_s = f_c \frac{E_s}{E_c} = \frac{29 \times 10^3 (f_c)}{3 \times 10^3} = 9.67 f_c$$

Substituting into the equilibrium equation:

$$3.14\,(9.76 f_c) + 141 f_c = 250$$

$$30.4 f_c + 141 f_c = 250$$

$$171.4 f_c = 250$$

$$f_c = 1.46\text{ ksi}$$

$$\therefore f_s = 9.67\,(1.46)\text{ ksi}$$

$$f_s = 14.1\text{ ksi}$$