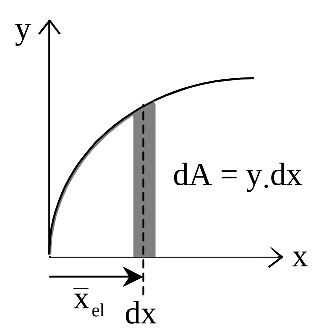
Moments of Inertia

Notation:

A = name for area
b = name for a (base) width
d = calculus symbol for differentiation
= name for a difference
= name for a depth
d_x = difference in the x direction between an area centroid ($\bar{x}$) and the centroid of the composite shape ($\hat{x}$)
d_y = difference in the y direction between an area centroid ($\bar{y}$) and the centroid of the composite shape ($\hat{y}$)
h = name for a height
$\bar{I}$ = moment of inertia about the centroid
I_c = moment of inertia about the centroid
I_x = moment of inertia with respect to an x-axis
I_y = moment of inertia with respect to a y-axis
J_o = polar moment of inertia, as is J
O = name for reference origin
r_o = polar radius of gyration
r_x = radius of gyration with respect to an x-axis
r_y = radius of gyration with respect to a y-axis
t_f = thickness of a flange
t_w = thickness of web of wide flange
x = horizontal distance
$\bar{x}$ = the distance in the x direction from a reference axis to the centroid of a shape
$\hat{x}$ = the distance in the x direction from a reference axis to the centroid of a composite shape
y = vertical distance
$\bar{y}$ = the distance in the y direction from a reference axis to the centroid of a shape
$\hat{y}$ = the distance in the y direction from a reference axis to the centroid of a composite shape
PL = plate symbol
$\int$ = symbol for integration
Σ = summation symbol

- The cross section shape and how it resists bending and twisting is important to understanding beam and column behavior.

- *Definition:* Moment of Inertia; the second area moment



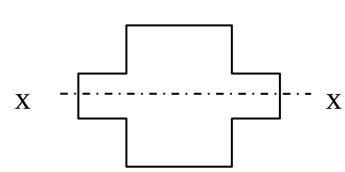
$$I_y = \int x^2 dA \qquad\qquad I_x = \int y^2 dA$$

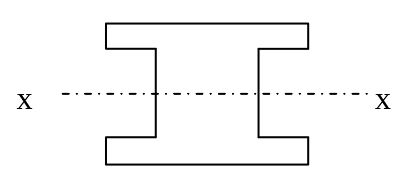
We can define a single integral using a narrow strip:

for I_x, strip is parallel to x   for I_y, strip is parallel to y

**I can be negative if the area is negative (a hole or subtraction).*

- A shape that has area at a greater distance away from an axis *through its centroid* will have a **larger** value of I.

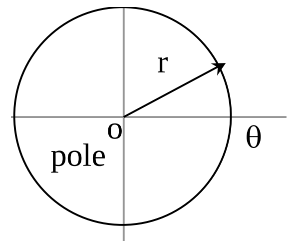




- Just like for center of gravity of an area, the moment of inertia can be determined with respect to *any* reference **axis**.

- *Definition:* Polar Moment of Inertia; the second area moment using polar coordinate axes

$$J_o = \int r^2 dA = \int x^2 dA + \int y^2 dA$$
$$J_o = I_x + I_y$$



- *Definition*: Radius of Gyration; the distance from the moment of inertia axis for an area at which the entire area could be considered as being concentrated at.

$$I_x = r_x{}^2 A \Rightarrow r_x = \sqrt{\frac{I_x}{A}}$$ radius of gyration in x

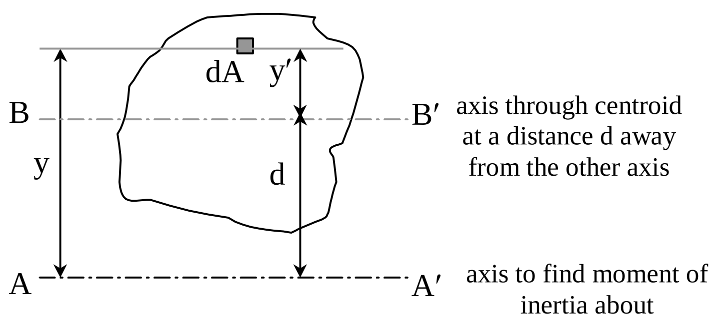
$$r_y = \sqrt{\frac{I_y}{A}}$$ radius of gyration in y

$$r_o = \sqrt{\frac{J_o}{A}}$$ polar radius of gyration, and $r_o{}^2 = r_x{}^2 + r_y{}^2$

The Parallel-Axis Theorem

- *The moment of inertia of an area with respect to any axis not through its centroid is equal to the moment of inertia of that area with respect to its own parallel centroidal axis plus the product of the area and the square of the distance between the two axes.*

$$I = \int y^2 dA = \int (y'\text{-}d)^2 dA$$
$$= \int y'^2 dA + 2d\int y' dA + d^2 \int dA$$



but $\int y' dA = 0$, because the centroid is on this axis, resulting in:

$I_x = I_{cx} + Ad_y{}^2$ (text notation) *or* $I_x = \bar{I}_x + Ad_y{}^2$

where I_{cx} (*or* $\bar{I}_x$) is the moment of inertia about the centroid of the area about an *x* axis and d_y is the y distance between the parallel axes

Similarly $I_y = \bar{I}_y + Ad_x^{\,2}$ Moment of inertia about a *y* axis

$J_o = \bar{J}_c + Ad^2$ Polar moment of Inertia

$r_o^2 = \bar{r}_c^2 + d^2$ Polar radius of gyration

$r^2 = \bar{r}^2 + d^2$ Radius of gyration

* *I can be negative again if the area is negative (a hole or subtraction).*
** *If* $\bar{I}$ is *not given in a chart, but* $\bar{x}$ *&* $\bar{y}$ *are: YOU MUST CALCULATE* $\bar{I}$ *WITH* $\bar{I} = I - Ad^2$

<u>Composite Areas</u>:

$I = \sum\bar{I} + \sum Ad^2$ where $\bar{I}$ is the moment of inertia about the centroid of the component area
d is the distance from the centroid of the component area to the centroid of the composite area (ie. $d_y = \hat{y} - \bar{y}$)

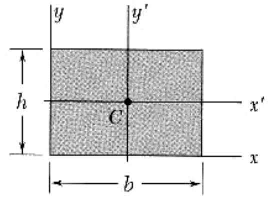
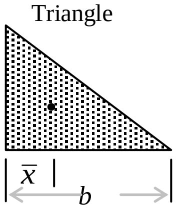
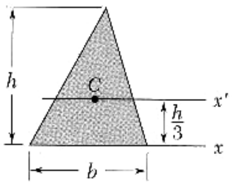
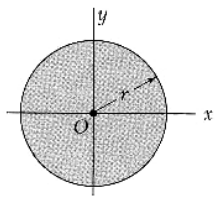
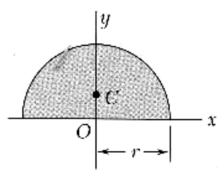
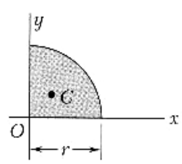
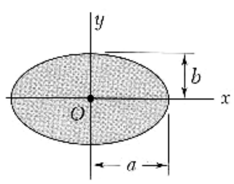
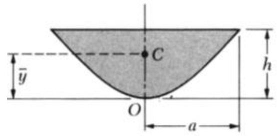
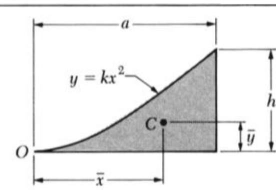
Basic Steps

1. Draw a reference origin.
2. Divide the area into basic shapes
3. Label the basic shapes (components)
4. Draw a table with headers of

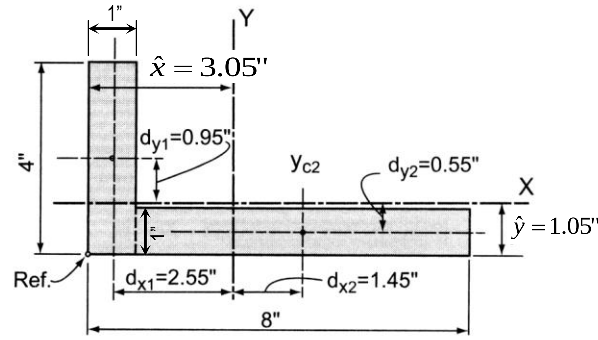
   Component, Area, $\bar{x}$, $\bar{x}A$, $\bar{y}$, $\bar{y}A$, $\bar{I}_x$, d_y, Ad_y^2, $\bar{I}_y$, d_x, Ad_x^2

5. Fill in the table values needed to calculate $\hat{x}$ and $\hat{y}$ for the composite
6. Fill in the rest of the table values.
7. Sum the moment of inertia ($\bar{I}$'s) and Ad^2 columns and add together.

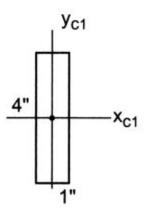
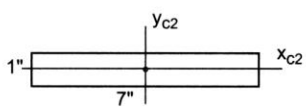
Geometric Properties of Areas

Rectangle		$\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$	Area = bh $\bar{x}$ = b/2 $\bar{y}$ = h/2
Triangle 		$\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{36}b^3h$	Area = $bh/2$ $\bar{x} = b/3$ $\bar{y} = h/3$
Circle		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$	Area = $\pi r^2 = \pi d^2/4$ $\bar{x}$ = 0 $\bar{y}$ = 0
Semicircle		$\bar{I}_x$ = 0.1098r^4 $\bar{I}_y$ = $\pi r^4/8$	Area = $\pi r^2/2 = \pi d^2/8$ $\bar{x}$ = 0 $\bar{y}$ = $4r/3\pi$
Quarter circle		$\bar{I}_x$ = 0.0549r^4 $\bar{I}_y$ = 0.0549r^4	Area = $\pi r^2/4 = \pi d^2/16$ $\bar{x}$ = $4r/3\pi$ $\bar{y}$ = $4r/3\pi$
Ellipse		$\bar{I}_x = \frac{1}{4}\pi ab^3$ $\bar{I}_y = \frac{1}{4}\pi a^3b$ $J_O = \frac{1}{4}\pi ab(a^2 + b^2)$	Area = πab $\bar{x}$ = 0 $\bar{y}$ = 0
Semiparabolic area Parabolic area		$\bar{I}_x$ = 16ah^3/175 $\bar{I}_y$ = 4a^3h/15	Area = $4ah/3$ $\bar{x}$ = 0 $\bar{y}$ = $3h/5$
Parabolic spandrel		$\bar{I}_x$ = 37ah^3/2100 $\bar{I}_y$ = a^3h/80	Area = $ah/3$ $\bar{x}$ = $3a/4$ $\bar{y}$ = $3h/10$

Example 1 (pg 257)



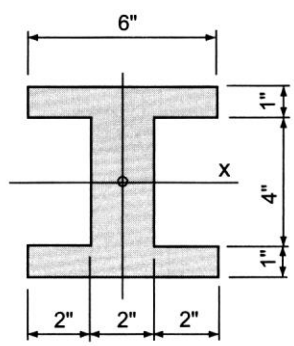
Find the moments of inertia ($\hat{x}$ = 3.05”, $\hat{y}$ = 1.05”).

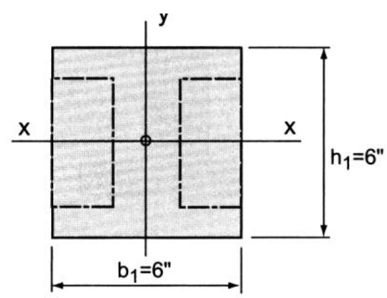
Component	I_{xc} (in.^{4})	d_y (in.)	Ad_y^2 (in.^{4})	I_{yc} (in.^{4})	d_x (in.)	Ad_x^2 (in.^{4})
	$\frac{(1)(4)^3}{12} = 5.33$	0.95	3.61	$\frac{(4)(1)^3}{12} = 0.33$	2.55	26.01
	$\frac{(7)(1)^3}{12} = 0.58$	0.55	2.12	$\frac{(1)(7)^3}{12} = 28.58$	1.45	14.72
	$\sum I_{xc} = 5.91$		$\sum Ad_y^2 = 5.73$	$\sum I_{yc} = 28.91$		$\sum Ad_x = 40.73$

Example 2 (pg 253)

Example Problem 7.6 (Figures 7.24 to 7.26)

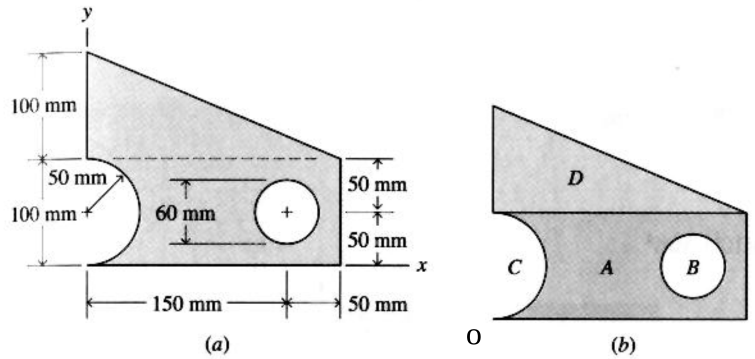
Determine the *I* about the centroidal *x*-axis.





Example 3

Determine the moments of inertia about the centroid of the shape.



Solution:

There is no reference origin suggested in figure (a), so the bottom left corner is good.

In figure (b) area A will be a complete rectangle, while areas C and A are "holes" with negative area and negative moment of inertias.

Area A = 200 mm x 100 mm = 20000 mm^2

I_x = (200 mm)(100 mm)^3/12 = 16.667 x 10^6 mm^4
I_y = (200 mm)^3(100 mm)/12 = 66.667 x 10^6 mm^4

Area B = -π(30 mm)^2 = -2827.4 mm^2

$I_x = I_y$ = -π (30 mm)^4/4 = -0.636 x 10^6 mm^4

Area C = -1/2π(50 mm)^2 = 3927.0 mm^2

I_x = -π (50 mm)^4/8 = -2.454 x 10^6 mm^4
I_y = -0.1098(50 mm)^4= -0.686 x 10^6 mm^4

Area D = 100 mm x 200 mm x 1/2 = 10000 mm^2

I_x = (200 mm)(100 mm)^3/36 = 5.556 x 10^6 mm^4
I_y = (200 mm)^3(100 mm)/36 = 22.222 x 10^6 mm^4

shape	A (mm^2)	$\bar{x}$ (mm)	$\bar{x}$A (mm^3)	$\bar{y}$ (mm)	$\bar{y}$A (mm^3)
A	20000	100	2000000	50	1000000
B	-2827.43	150	-424115	50	-141372
C	-3926.99	21.22066	-83333.3	50	-196350
D	10000	66.66667	666666.7	133.3333	1333333
	23245.58		2159218		1995612

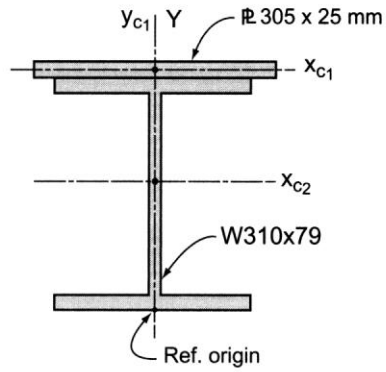
$$\hat{x} = \frac{2159218\text{mm}^3}{23245.58\text{mm}^2} = 92.9\text{ mm}$$

$$\hat{y} = \frac{1995612\ \text{mm}^3}{23245.58\ \text{mm}^2} = 85.8\text{ mm}$$

shape	I_x (mm^4)	d_y (mm)	Ad_y^2 (mm^4)	I_y (mm^4)	d_x (mm)	Ad_x^2 (mm^4)
A	16666667	35.8	25632800	66666667	-7.1	1008200
B	-636173	35.8	-3623751.73	-636173	-57.1	-9218592.093
C	-2454369	35.8	-5032988.51	-686250	71.67934	-20176595.22
D	5555556	-47.5333	22594177.8	22222222	26.23333	6881876.029
	19131680		39570237.5	87566466		-21505111.29

So, I_x = 19131680 + 39570237.5 = 58701918 = 58.7 x 10^6 mm^4

I_x = 87566466 +-21505111.3 = 43572025 = 66.1 x 10^6 mm^4



Example 4 (pg 258)

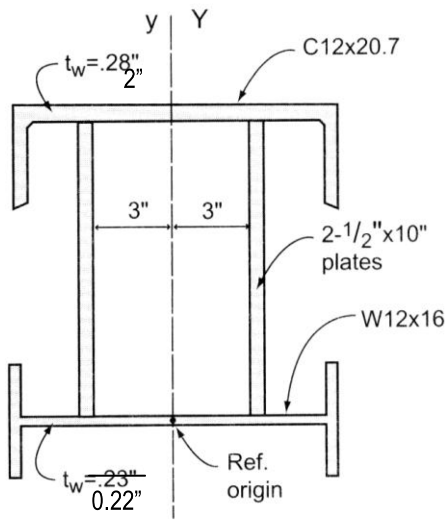
Example Problem 7.10 (Figures 7.35 and 7.36)

Locate the centroidal x and y axes for the cross-section shown. Use the reference origin indicated and assume that the steel plate is centered over the flange of the wide-flange section. Compute the I_x and I_y about the major centroidal axes.

Example 5 (pg 249)*

Example Problem 7.5 (Figures 7.16 and 7.17)

A composite or built-up cross-section for a beam is fabricated using two $\frac{1}{2}$" × 10" vertical plates with a C12 × 20.7 channel section welded to the top and a W12 × 16 section welded to the bottom as shown. Determine the location of the major x-axis using the center of the W12 × 16's web as the reference origin. Also determine the moment of inertia about both major centroidal axes.



shape	A (in²)	$\bar{x}$ (in)	$\bar{x}$A (in³)	$\bar{y}$ (in)	$\bar{y}$A (in³)
channel	6.09	0	0.00	9.694	59.04
left plate	5	-3.25	-16.25	5.11	25.55
right plate	5	3.25	16.25	5.11	25.55
wide flange	4.71	0	0.00	0	0.00
	20.80		0.00		110.14

$$\hat{x} = \frac{0\ in^3}{20.8\ in^2} = 0\ in$$

$$\hat{y} = \frac{110.14\ in^3}{20.8\ in^2} = 5.295\ in$$

shape	I_x (in⁴)	d_y (in)	Ad_y^2 (in⁴)	I_y (in⁴)	d_x (in)	Ad_x^2 (in⁴)
channel	3.880	-4.399	117.849	129.000	0.000	0.000
left plate	41.667	0.185	0.171	0.104	3.250	52.813
right plate	41.667	0.185	0.171	0.104	-3.250	52.813
wide flange	2.800	5.295	132.054	103.000	0.000	0.000
	90.013		250.245	232.208		105.625

$I_x = 90.013 + 250.245 = 340.259 = 340.3\ in^4$

$I_y = 232.208 + 105.625 = 337.833 = 337.8\ in^4$