Beam Design and Deflections

Notation:

a = name for width dimension
A = name for area
$A_{req'd-adj}$ = area required at allowable stress when shear is adjusted to include self weight
A_{web} = area of the web of a wide flange section
b = width of a rectangle
= total width of material at a horizontal section
= name for height dimension
c = largest distance from the neutral axis to the top or bottom edge of a beam
c_1 = coefficient for shear stress for a rectangular bar in torsion
d = calculus symbol for differentiation
DL = shorthand for dead load
E = modulus of elasticity
f_b = bending stress
f_p = bearing stress (see P)
f_v = shear stress
f_{v-max} = maximum shear stress
F_b = allowable bending stress
F_v = allowable shear stress
F_p = allowable bearing stress
F_y = yield strength
F_{yweb} = yield strength of the web material
h = height of a rectangle
I = moment of inertia with respect to neutral axis bending
I_{trial} = moment of inertia of trial section
$I_{req'd}$ = moment of inertia required at limiting deflection
J = polar moment of inertia
L = name for span length
LL = shorthand for live load
$LRFD$ = load and resistance factor design
M = internal bending moment
M_{max} = maximum internal bending moment
$M_{max-adj}$ = maximum bending moment adjusted to include self weight
M_n nominal flexure strength with the full section at the yield stress for LRFD
M_u = maximum moment from factored loads for LRFD
P = name for axial force vector
Q = first moment area about a neutral axis
R = radius of curvature of a deformed beam
S = section modulus
$S_{req'd}$ = section modulus required at allowable stress
T = torque (axial moment)
V = internal shear force
V_{max} = maximum internal shear force
$V_{max-adj}$ = maximum internal shear force adjusted to include self weight
V_u = maximum shear from factored loads for LRFD
w = name for distributed load
$w_{self\ wt}$ = name for distributed load from self weight of member
x = horizontal distance
y = vertical distance
Δ_{actual} = actual beam deflection
$\Delta_{allowable}$ = allowable beam deflection
Δ_{limit} = allowable beam deflection limit
Δ_{max} = maximum beam deflection
ϕ_b = resistance factor for flexure in LRFD design
ϕ_v = resistance factor for shear for LRFD
γ = density or unit weight
θ = slope of the beam deflection curve
ρ = radial distance
$\int$ = symbol for integration
Σ = summation symbol

Criteria for Design

Allowable bending stress or bending stress from LRFD should not be exceeded:

$$F_b \geq f_b = \frac{Mc}{I}$$

Knowing M and F_b, the minimum section modulus fitting the limit is:

$$S_{req'd} \geq \frac{M}{F_b}$$

Besides strength, we also need to be concerned about *serviceability*. This involves things like limiting deflections & cracking, controlling noise and vibrations, preventing excessive settlements of foundations and durability. When we know about a beam section and its material, we can determine beam deformations.

Determining Maximum Bending Moment

Drawing V and M diagrams will show us the maximum values for design. Remember:

$$V = \Sigma(-w)dx \qquad \frac{dV}{dx} = -w \qquad \frac{dM}{dx} = V$$

$$M = \Sigma(V)dx$$

Determining Maximum Bending Stress

For a prismatic member (constant cross section), the maximum normal stress will occur at the maximum moment.

For a *non-prismatic* member, the stress varies with the cross section AND the moment.

Deflections

If the bending moment changes, M(x) across a beam of constant material and cross section then the curvature will change:

$$\frac{1}{R} = \frac{M(x)}{EI}$$

The slope of the n.a. of a beam, θ, will be tangent to the radius of curvature, R:

$$\theta = slope = \frac{1}{EI}\int M(x)dx$$

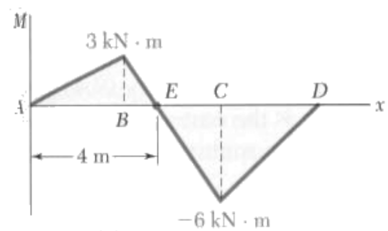
The equation for deflection, y, along a beam is:

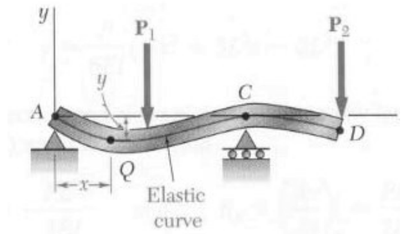
$$y = \frac{1}{EI}\int \theta dx = \frac{1}{EI}\iint M(x)dx$$

Elastic curve equations can be found in handbooks, textbooks, design manuals, etc...Computer programs can be used as well.

Elastic curve equations can be **superpositioned** ONLY if the stresses are in the elastic range.

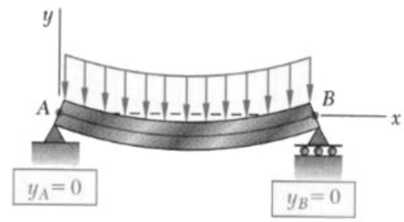
The deflected shape is roughly the shame shape as the bending moment diagram flipped but is constrained by supports and geometry.





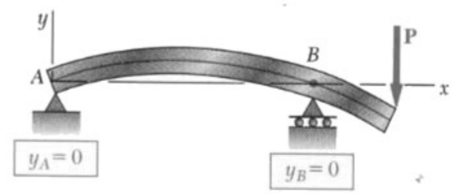
Boundary Conditions

The boundary conditions are geometrical values that we know – slope or deflection – which may be restrained by supports or symmetry.



(a) Simply supported beam

At Pins, Rollers, Fixed Supports: $y = 0$



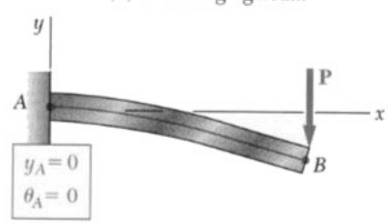
(b) Overhanging beam

At Fixed Supports: $\theta = 0$

At Inflection Points From Symmetry: $\theta = 0$

The Slope Is Zero At The Maximum Deflection y_{max}:.

$$\theta = \frac{dy}{dx} = slope = 0$$



(c) Cantilever beam

Allowable Deflection Limits

All building codes and design codes limit deflection for beam types and damage that could happen based on service condition and severity.

$$y_{max}(x) = \Delta_{actual} \le \Delta_{allowable} = {}^{L}\!/_{value}$$

Use	LL only	DL+LL
Roof beams:		
Industrial	L/180	L/120
Commercial		
plaster ceiling	L/240	L/180
no plaster	L/360	L/240
Floor beams:		
Ordinary Usage	L/360	L/240
Roof or floor (damageable elements)		L/480

Beam Loads & Load Tracing

In order to determine the loads on a beam (or girder, joist, column, frame, foundation...) we can start at the top of a structure and determine the *tributary area* that a load acts over and the beam needs to support. Loads come from material weights, people, and the environment. This area is assumed to be from half the distance to the next beam over to halfway to the next beam.

The reactions must be supported by the next lower structural element *ad infinitum*, to the ground.

Design Procedure

The intent is to find the most light weight member satisfying the section modulus size.

1. Know F_b (allowable stress) for the material or F_y & F_u for LRFD.
2. Draw V & M, finding M_{max}.
3. Calculate $S_{req'd}$. This step is equivalent to determining $f_b = \frac{M_{max}}{S} \leq F_b$
4. For rectangular beams $S = \frac{bh^2}{6}$
   - For steel or timber: use the section charts to find S that will work *and remember that the beam self weight will increase $S_{req'd}$.* And for steel, the design charts show the lightest section within a grouping of similar S's. $w_{self\ wt} = \gamma A$
   - For any thing else, try a nice value for b, and calculate h or the other way around.

*****Determine the "updated" V_{max} and M_{max} including the beam self weight, and verify that the updated $S_{req'd}$ has been met.*******

5. Consider lateral stability

6. Evaluate horizontal shear stresses using V_{max} to determine if $f_v \leq F_v$

   For rectangular beams, W's, and others: $$f_{v-max} = \frac{3V}{2A} \approx \frac{V}{A_{web}} \ or\ \frac{VQ}{Ib}$$

7. Provide adequate bearing area at supports: $$f_p = \frac{P}{A} \leq F_p$$

8. Evaluate shear due to torsion $$f_v = \frac{T\rho}{J} \ or\ \frac{T}{c_1 ab^2} \leq F_v$$
   (circular section or rectangular)

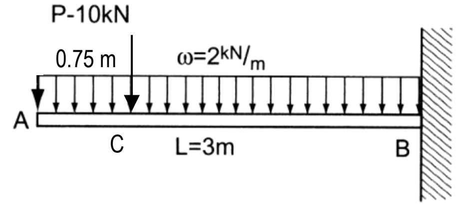
9. Evaluate the deflection to determine if $\Delta_{maxLL} \leq \Delta_{LL-allowed}$ and/or $\Delta_{maxTotal} \leq \Delta_{T-allowed}$

**** *note: when $\Delta_{calculated} > \Delta_{limit}$, $I_{required}$ can be found with: and $S_{req'd}$ will be satisfied for similar self weight* ***** $$I_{req'd} \geq \frac{\Delta_{too\ big}}{\Delta_{limit}} I_{trial}$$

FOR ANY EVALUATION:

Redesign (with a new section) at any point that a stress or serviceability criteria is NOT satisfied and re-evaluate each condition until it is satisfactory.

Example 1 (*changed from* pg 284) (*superpositioning*)



Example Problem 8.5 (Semi-Graphical Method)

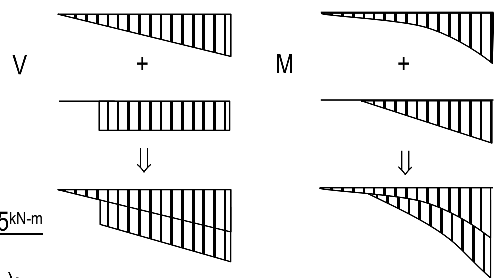
A cantilever beam supports a uniform load of $\omega = 2\,^{kN}/_{m}$ over its entire span, plus a concentrated load of 10 kN at the free end. *Investigate using Beam Diagrams and Formulas.*

SOLUTION:

By examining the support conditions, we are looking for a cantilevered beam from Cases 18 through 23.

There is a case for uniformly distributed load across the span (19) and for a load at any point (21).

For both these cases, it shows that the maximum shear AND maximum moment are located at the fixed end. If we add the values, the shear and diagrams should look like this:



We can find the maximum shear (at B) from V = P + wl = $10^{kN} + 2^{kN/m}\cdot 3^{m}$ = <u>16^{kN}</u>

The maximum moment (at B) will be M = $Pb + wl^2/2 = 10^{kN}\cdot 2.25^{m} + 2^{kN/m}(3^{m})^2/2$ = <u>$31.5^{kN\text{-}m}$</u>

The key values for the diagrams can be found with the general equations (V_x and M_x):

$V_{C\leftarrow} = -wx = -2^{kN/m}\cdot 0.25^{m} = -0.5^{kN}$; $V_{C\rightarrow} = -P - wx = -10^{kN} - 2^{kN/m}\cdot 0.25^{m} = -10.5^{kN}$

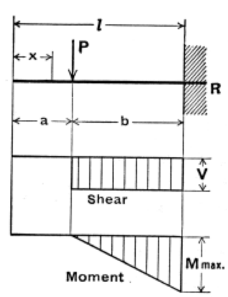
$M_C = -wx^2/2 = 2^{kN/m}(0.25^{m})^2/2 = 0.0625^{kN\text{-}m}$

We can find the maximum deflection by looking at the cases. Both say $\Delta_{max.}$ (at free end), so the values can be added directly. *Superpositioning of values <u>must</u> be at the same x location.* Assume E = 70×10^3 MPa and I = 45×10^6 mm^4

$$\Delta_{total} = \frac{Pb^2}{6EI}(3l-b) + \frac{wl^4}{8EI} = \frac{10^{kN}(2.25m)^2(10^3\,{}^{mm}/_{m})^3(10^3\,{}^{N}/_{kN})}{6(70\times10^3 MPa)(45\times10^6 mm^4)}(3\cdot 3m - 2.25m) + \frac{2^{kN}/_{m}(3m)^4(10^3\,{}^{mm}/_{m})^3(10^3\,{}^{N}/_{kN})}{8(70\times10^3 MPa)(45\times10^6 mm^4)}$$

= 18.08 mm + 6.43 mm = 24.5 mm

21. CANTILEVER BEAM—CONCENTRATED LOAD AT ANY POINT



Total Equiv. Uniform Load . . . . = $\frac{8Pb}{l}$

R = V . . . . = P

M max. (at fixed end) . . . . . = Pb

M_x (when x > a) . . . . . = P (x — a)

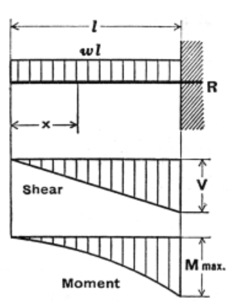
Δmax. (at free end) . . . . . = $\frac{Pb^2}{6EI}(3l - b)$

Δa (at point of load) . . . . = $\frac{Pb^3}{3EI}$

Δx (when x < a) . . . . . = $\frac{Pb^2}{6EI}(3l - 3x - b)$

Δx (when x > a) . . . . . = $\frac{P(l-x)^2}{6EI}(3b - l + x)$

19. CANTILEVER BEAM—UNIFORMLY DISTRIBUTED LOAD



Total Equiv. Uniform Load . . . . = $4wl$

R = V . . . . = wl

V_x . . . . = wx

M max. (at fixed end) . . . . . = $\frac{wl^2}{2}$

M_x . . . . = $\frac{wx^2}{2}$

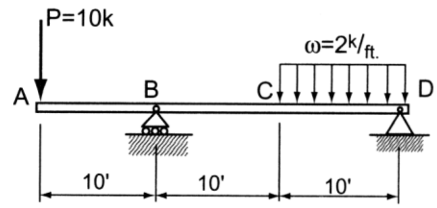
Δmax. (at free end) . . . . . = $\frac{wl^4}{8EI}$

Δx . . . . = $\frac{w}{24EI}(x^4 - 4l^3x + 3l^4)$

Example 2 (pg 275) (*superpositioning*)

Example Problem 8.2(Equilibrium Method)

Draw V and M diagrams for an overhang beam (Figure 8.12) loaded as shown. Determine the critical V_{max} and M_{max} locations and magnitudes *using Beam Diagrams and Formulas.*

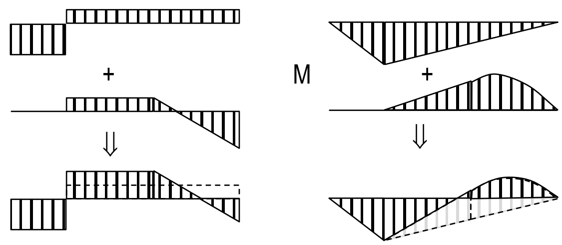


SOLUTION:

By examining the support conditions, we are looking for beam with an overhang on one end from Cases 24 through 28. (Even though the overhang is on the right, and not on the left like our beam, we can still use the information by recognizing that we can mirror the figure about the left end.)

There is a case for a load at the end (26) but none for a load in between the supports. This is because it behaves exactly like a simply supported beam in this instance (no shear or bending on the overhang). The case for this is #5 (reversed again).

If we "flip" the diagrams (both vertically and horizontally) and add the values, the resulting shear and bending moment should look like this:



We still have to find the peak values of shear and the location of the zero shear to find the critical moment values.

(Notice R_1 is shown *down*.)

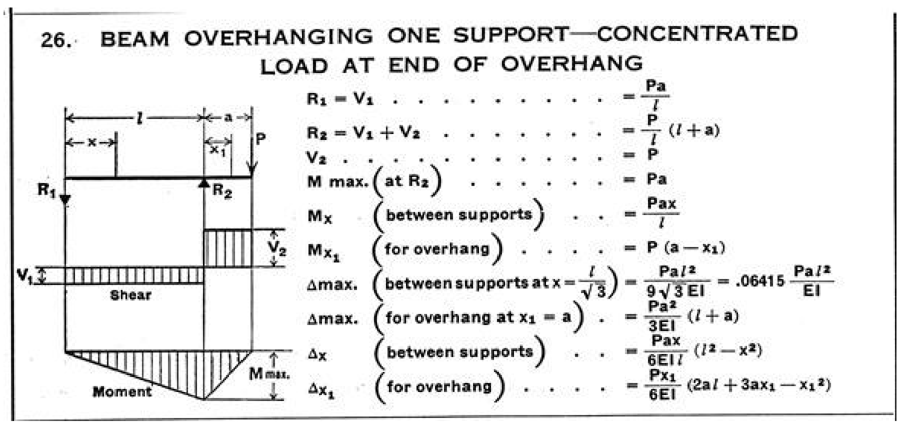
$$R_{1(D)} = -\frac{Pa}{l} + \frac{wa}{2l}(2l-a) = -\frac{10^k \cdot 10^{ft}}{20^{ft}} + \frac{2^{k/ft} \cdot 10^{ft}}{2 \cdot 20^{ft}}\left(2 \cdot 20^{ft} - 10^{ft}\right) = 10^k$$

$$R_{2(B)} = 10^{ft} + 2^{k/ft} \cdot 10^{ft} - 10^k = 20^k$$ (from the total downward load – $R_{1(D)}$)

$V_A = -10^k$ $V_B = -10^k + 20^k = 10^k$
$V_D = 10^k – 2^{k/ft}(10ft) = -10^k$
x (from B) = $10^k/2^{k/ft}$ = 5 ft
$M_B = -10^k(10^{ft}) = -100^{k\text{-}ft}$
($M_C = -100^{k\text{-}ft} + 10^k(10^{ft}) = 0$)
$M_x = 0 + 10^{k/ft}(5^{ft})/2 = 25^{k\text{-}ft}$

$V_{MAX} = 10^k$ $M_{MAX} = -100^{k\text{-}ft}$

We can calculate the deflection between the supports. (And at the end for case 5 if we derive the slope!) Assume E = 29x10³ ksi and I = 103 in⁴



26. BEAM OVERHANGING ONE SUPPORT—CONCENTRATED LOAD AT END OF OVERHANG

$R_1 = V_1$ . . . . $= \frac{Pa}{l}$
$R_2 = V_1 + V_2$ . . . . $= \frac{P}{l}(l+a)$
V_2 . . . . $= P$
M max. (at R_2) . . . . $= Pa$
M_x (between supports) . . . $= \frac{Pax}{l}$
M_{x_1} (for overhang) . . . . $= P(a - x_1)$
Δmax. (between supports at $x = \frac{l}{\sqrt{3}}$) $= \frac{Pal^2}{9\sqrt{3}EI} = .06415\frac{Pal^2}{EI}$
Δmax. (for overhang at $x_1 = a$) . $= \frac{Pa^2}{3EI}(l+a)$
Δx (between supports) . . $= \frac{Pax}{6EIl}(l^2 - x^2)$
Δx_1 (for overhang) . . . . $= \frac{Px_1}{6EI}(2al + 3ax_1 - x_1^2)$

We'll investigate the maximum between the supports from case 26 (because it isn't obvious where the maximum will be.)

$x = l/\sqrt{3} = 20^{ft}/\sqrt{3} = 11.55^{ft}$ (to left of D) and $\Delta_{total} = \Delta_{max.-case\ 26} + \Delta_{x-case\ 5}$ with x (11.55ft) greater than a (10ft):

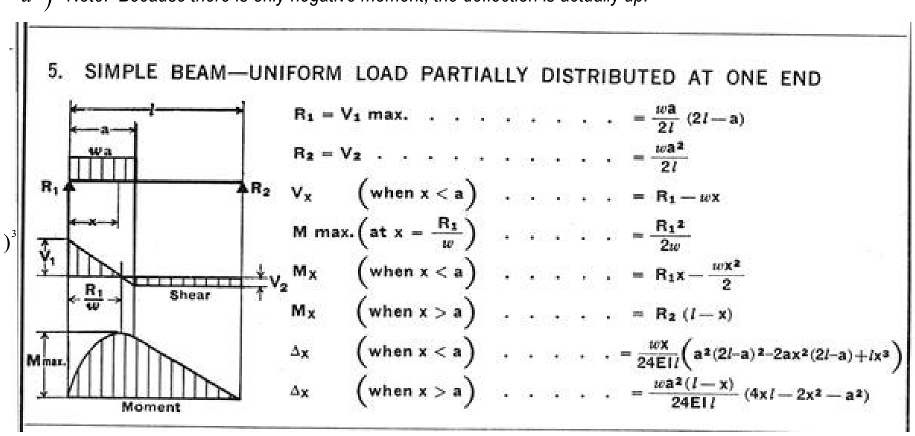
$$\Delta_{total} = .06415\frac{Pal^2}{EI} + \frac{wa^2(l-x)}{24EIl}\left(4xl - 2x^2 - a^2\right)$$ *Note: Because there is only negative moment, the deflection is actually up!*

$$= -.06415\frac{10^k(10^{ft})(20^{ft})^2(12^{in/ft})^3}{(29 \cdot 10^3 ksi)(103in^4)} +$$

$$\frac{2^{k/ft}(10^{ft})^2(20^{ft} - 10^{ft})}{24(29 \cdot 10^3 ksi)(103in^4)(20^{ft})} \times \ldots$$

$$\left(4 \cdot 11.55^{ft} \cdot 20^{ft} - 2(11.55^{ft})^2 - (10^{ft})^2\right)(12^{in/ft})^3$$

= -1.484 in + 1.343 in = -0.141 in (up)



5. SIMPLE BEAM—UNIFORM LOAD PARTIALLY DISTRIBUTED AT ONE END

$R_1 = V_1$ max. . . . . $= \frac{wa}{2l}(2l - a)$
$R_2 = V_2$ . . . . $= \frac{wa^2}{2l}$
V_x (when $x < a$) . . . . . $= R_1 - wx$
M max. (at $x = \frac{R_1}{w}$) . . . . . $= \frac{R_1^2}{2w}$
M_x (when $x < a$) . . . . . $= R_1x - \frac{wx^2}{2}$
M_x (when $x > a$) . . . . . $= R_2(l - x)$
Δx (when $x < a$) . . . . . $= \frac{wx}{24EIl}\left(a^2(2l-a)^2 - 2ax^2(2l-a) + lx^3\right)$
Δx (when $x > a$) . . . . . $= \frac{wa^2(l-x)}{24EIl}(4xl - 2x^2 - a^2)$

Beam Design Flow Chart